Ultracold atoms and quantum simulators
Problem set 6

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1 Collective modes of quantum gases

In this problem, we will calculate the frequency of some low-energy collective oscillation modes of quantum gases held in a harmonic trap. The N-particle system is described by a many-body hamiltonian

$$\hat{\mathcal{H}} = \sum_{i=1}^{N} \left(\frac{\hat{\boldsymbol{p}}_i^2}{2m} + \frac{m\omega_r^2}{2}\, \hat{\boldsymbol{r}}_i^2 \right) + \hat{\mathcal{V}}_{\text{int}} \tag{1}$$

which contains the isotropic harmonic trapping potential.

The eigenstates of the hamiltonian are denoted $|n\rangle$: $\hat{\mathcal{H}}|n\rangle = E_n|n\rangle$, $\hbar\omega_n = E_n$ so that one has $\hat{\mathcal{H}} = \sum_n E_n|n\rangle\langle n|$. We note $|0\rangle$ the ground state of, for simplicity, we take E_0 as the reference energy in the following: $E_0 = 0$. The goal is to find the energy of the first excited states. We will show that, provided we know the equation of state of the many-body ground state, we can extract the frequency of the low-lying excitations or in turn measuring the frequency of these excitations provides information on the nature of the ground state.

1) Consider the hermitian operator $\hat{\mathcal{F}}$, we define the following "sum rules":

$$m_p = \sum_n (\hbar\omega_n)^p \; |\langle n|\hat{\mathcal{F}}|0\rangle|^2 \tag{2}$$

Show that the frequency of the lowest energy mode n_0 that has a non-zero coupling to the ground state through $\hat{\mathcal{F}}$ ($\langle 0|\hat{\mathcal{F}}|n_0\rangle \neq 0$) can be bounded by:

$$(\hbar\omega_{n_0})^2 \leqslant \frac{m_1}{m_{-1}} \tag{3}$$

and that this is a strict equality if the operator $\hat{\mathcal{F}}$ couples only to $|n_0\rangle$: $\langle 0|\hat{\mathcal{F}}|n\rangle = \delta_{n,n_0}$.

2) Show that

$$m_1 = \frac{1}{2} \left\langle \left[\hat{\mathcal{F}}, \left[\hat{\mathcal{H}}, \hat{\mathcal{F}}\right]\right] \right\rangle \tag{4}$$

where $\langle \cdots \rangle$ denotes the expectation value in the ground state.

3) Consider the the perturbed hamiltonian

$$\hat{\mathcal{H}}' = \hat{\mathcal{H}} - \lambda\hat{\mathcal{F}} \tag{5}$$

where λ is taken small so that we can consider the second part as a small perturbation. We write the states of the perturbed hamiltonian $|n'\rangle$ Using perturbation theory show that

$$\langle \hat{\mathcal{F}} \rangle' = \langle \hat{\mathcal{F}} \rangle + 2\lambda\, m_{-1} \tag{6}$$

where $\langle \cdots \rangle'$ denotes the expectation value in the ground state of $\hat{\mathcal{H}}'$.

4) We will now calculate the frequency of the "breathing mode" in an isotropic harmonic trap. For this, we use the operator

$$\hat{\mathcal{F}} = \sum_i \hat{\boldsymbol{r}}_i^2, \tag{7}$$

prove that

$$m_1 = \frac{2\hbar^2 N}{m} \langle r^2 \rangle. \tag{8}$$

5) Show that if $\frac{2\lambda}{m} \ll \omega_r^2$

$$\langle r^2 \rangle' = \langle r^2 \rangle - \frac{2\lambda}{m} \frac{\partial \langle r^2 \rangle}{\partial \omega_r^2} \tag{9}$$

so that

$$m_{-1} = -\frac{N}{m} \frac{\partial \langle r^2 \rangle}{\partial \omega_r^2}. \tag{10}$$

We assume that $\hat{\mathcal{F}}$ couples only to one mode, so that we have

$$\omega_{n_0}^2 = -2\langle r^2 \rangle \Bigg/ \frac{\partial \langle r^2 \rangle}{\partial \omega_r^2}\ . \tag{11}$$

This expressions shows that we are calculating the frequency of a collective breathing mode in which the overall radius of the cloud oscillates with time in the harmonic trap. To calculate this frequency, we will now make use of the local density approximation:

$$\mu(\boldsymbol{r}) = \mu_0 - V_{\text{ext}}(\boldsymbol{r}) \tag{12}$$

We recall the equation of state for a homogeneous Bose-Einstein condensate and a Fermi gas: $\mu = g\, n$ with $g = \frac{4\pi\hbar^2 a}{m}$, and $\mu = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}$. To treat both cases, we will assume a general expression for the EoS: $n \propto \mu^p$.

6) By using $N = \int_V d\boldsymbol{r}\, n(\boldsymbol{r})$, show that

$$\mu_0 \propto \omega_r^{\frac{3}{p+3/2}} \tag{13}$$

and that

$$\langle r^2 \rangle \propto \omega_r^{-\frac{2p}{2p+3}} \tag{14}$$

7) Conclude finally that the frequency is

$$\omega_{\text{breathing}} = \omega_r \sqrt{\frac{2p+3}{2p}}. \tag{15}$$

What is it for a BEC? And for a Fermi gas?
We thus see that the precise measurement of collective oscillation frequencies in a harmonic trap allows to draw information on the equation of state of a homogenous system!
Next we consider an axially symmetric external trap: $V_{\text{ext}} = \frac{m\omega_\perp^2}{2}\rho^2 + \frac{m\omega_z^2}{2}z^2$ and we define the mean trap frequency $\bar{\omega} = (\omega_\perp^2 \omega)^{1/3}$. We will calculate the frequency of the axial breathing mode: an oscillation of the z radius of the cloud. The right operator to use is thus $\hat{\mathcal{F}} = \sum_i \hat{\boldsymbol{z}}_i^2$.

8) Applying the same method as above, show that

$$\omega_{n_0}^2 = -2\langle z^2 \rangle \Bigg/ \frac{\partial \langle z^2 \rangle}{\partial \omega_z^2} \tag{16}$$

$$\omega_{\text{axial breathing}} = \omega_z \sqrt{\frac{2p+3}{p+1}}. \tag{17}$$

Thus for a BEC the mode frequency is $\sqrt{5/2}\,\omega_z$ while for a Fermi gas it is $\sqrt{12/5}\,\omega_z$. The precise measurement of the frequency of this mode represented in the figure below has allowed a remarkable test of the change of nature of the ground state of a spin-1/2 Fermi gas in the BEC to BCS crossover. On the BEC side, the frequency was found to approach the value predicted for a BEC (dashed line at $\sqrt{5/2}$) which shows that the Fermi gas does indeed form a BEC of molecules, while at unitarity ($1/k_{\text{F}}a = 0$) where the gas is supposed to have the same form of EoS as a Fermi gas, one indeed finds the Fermi gas prediction (star at $\sqrt{12/5}$).

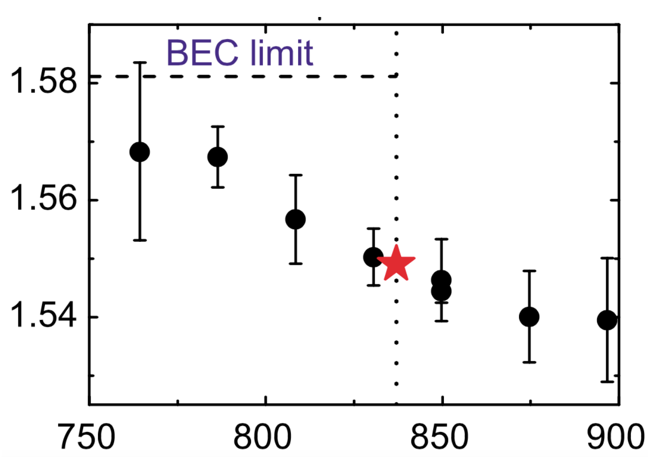

Figure 1: Frequency of the axial breathing mode in a cylindrical harmonic trap in the BEC-BCS crossover. The vertical axis is $\omega_{\text{axial breathing}}/\omega_z$ and the horizontal axis is the magnetic field in Gauss. The position of the resonance where one obtains a unitary Fermi gas $1/k_{\text{F}}a = 0$ is indicated by the dashed line.