Ultracold atoms and quantum simulators
Problem set 4

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1 Hydrodynamic equations

Here we will show that the Gross-Pitaevskii equation can be recast in the form of hydrodynamic equations. We recall the time-dependent Gross-Pitaevskii equation (GPE) for a homogeneous system:

$$i\hbar\,\partial_t\psi(\boldsymbol{r},t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\boldsymbol{r},t) + g\,|\psi(\boldsymbol{r},t)|^2\psi(\boldsymbol{r},t). \tag{1}$$

1) By writing ψ in the amplitude-phase form: $\psi = \sqrt{n}\,e^{i\phi}$, show that the GPE leads to two hydrodynamic equation:

$$\partial_t n + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = 0 \tag{2}$$

$$m\partial_t\boldsymbol{v} + \boldsymbol{\nabla}\left(\frac{mv^2}{2} + gn - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) = 0 \tag{3}$$

where you will give the expression of $\boldsymbol{v}$. Can you comment on these equations?

2 The Bogoliubov dispersion relation

The ground state solution of eqs. (2, 3) is simply $n(\boldsymbol{r}) = n_0$ ($\boldsymbol{\nabla}n_0 = 0$), $\boldsymbol{v} = 0$.

1) By taking small perturbations of n and $\boldsymbol{v}$ around the ground state solution: $n = n_0 + \delta n$, $\boldsymbol{v} = 0 + \delta\boldsymbol{v}$ and linearizing eqs. (2, 3), show that we can obtain:

$$\partial_t\delta n + n_0\,\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \tag{4}$$

$$m\,\partial_t\boldsymbol{v} + \boldsymbol{\nabla}(g\delta n - \frac{\hbar^2}{4\,mn_0}\nabla^2\delta n) = 0 \tag{5}$$

2) Taking the plane wave ansantz $\delta n = \eta\,e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, $\delta\boldsymbol{v} = \boldsymbol{\gamma}\,e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})}$, derive the Bogoliubov dispersion relation

$$\omega = \sqrt{\frac{k^2}{2m}\left(2\,gn_0 + \frac{\hbar^2k^2}{2m}\right)} \tag{6}$$

3) What happens when $g < 0$?
We find a mechanical instability of the Bose-Einstein condensate, which tends to collapse on itself if $g < 0$, this is why one usually says that interactions are effectively attractive in that regime.

3 Dipolar Bose-Einstein condensates

When the atomic species cooled-down to Bose-Einstein condensation has a strong magnetic moment $\boldsymbol{\mu}$, the dipole-dipole interaction between the atoms arising from their strong magnetic dipole has to be taken into account. This dipole-dipole interaction reads:

$$V_{\mathrm{dd}}(\boldsymbol{r}) = \frac{\mu_0 \mu^2}{4\pi r^3}\left(1 - 3\cos^2\theta\right), \tag{7}$$

where it is assumed that the dipoles are all aligned along the z axis, and that $\boldsymbol{r}$ makes an angle θ with this axis, μ_0 is the vacuum permeability. The particularity of this interaction is to be non-local (long range) and anisotropic: aligned dipoles attract each other when they are head-to-tails and repel when side-by-side. One can show that the GPE acquires an extra non-local interaction term:

$$i\hbar\,\partial_t\psi(\boldsymbol{r},t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\boldsymbol{r},t) + g\,|\psi|^2\,\psi(\boldsymbol{r},t) + \psi(\boldsymbol{r},t)\int V_{\mathrm{dd}}(\boldsymbol{r}-\boldsymbol{r}')\left|\psi(\boldsymbol{r}',t)\right|^2 d\boldsymbol{r}'. \tag{8}$$

For simpliciy, we write the last term $\psi(\boldsymbol{r},t)\Phi_{\mathrm{dd}}$.

1) Show that the hydrodynamic equations remain the same except for an extra term in (3):

$$\boldsymbol{\nabla}\left(\Phi_{\mathrm{dd}}(\boldsymbol{r})\right) \tag{9}$$

2) Show that $\Phi^0_{\mathrm{dd}} = \int d\boldsymbol{r}' V_{\mathrm{dd}}(\boldsymbol{r}-\boldsymbol{r}')n_0 = 0$.

3) Conclude that the linearization of the hydrodynamic equations lead to the following form in the presence of dipolar interactions:

$$\partial_t\delta n + n_0\,\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \tag{10}$$

$$m\,\partial_t\boldsymbol{v} + \boldsymbol{\nabla}\left(g\delta n + \int V_{\mathrm{dd}}(\boldsymbol{r}-\boldsymbol{r}')\delta n(\boldsymbol{r}')d\boldsymbol{r}' - \frac{\hbar^2}{4\,mn_0}\nabla^2\delta n\right) = 0, \tag{11}$$

4) Show that when using the plane wave ansatz, the extra term due to the dipole-dipole interaction is simply

$$\boldsymbol{\nabla}\left(\eta\,e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}\,\tilde{V}_{\mathrm{dd}}(\boldsymbol{k})\right), \tag{12}$$

where $\tilde{V}_{\rm dd}(\boldsymbol{k})$ is the Fourier transform of the dipole dipole interaction.

4) We give below the expression of $\tilde{V}$, it is independent on the magnitude of $\boldsymbol{k}$ and depends only on the angle α between $\boldsymbol{k}$ and the magnetic field

$$\tilde{V}_{\rm dd}(\boldsymbol{k}) = \frac{\mu_0\mu^2}{3}(3\cos^2\alpha - 1). \tag{13}$$

Derive the dispersion relation for the elementary excitations in a dipolar Bose-Einstein condensate:

$$\omega = \sqrt{\frac{k^2}{2m}\left(2\,gn_0(1+\varepsilon_{\rm dd}(3\cos^2\alpha - 1)) + \frac{\hbar^2k^2}{2m}\right)} \tag{14}$$

5) We thus find that the speed of sound depends on the direction of propagation with respect to the magnetic field. What happens when $\varepsilon_{\rm dd} > 1$?

Since the speed of sound is anisotropic, so is the critical velocity for superfluidity according to Landau's argument. This has been tested by moving a laser beam through a dipolar Bose-Einstein condensate, and indeed the critical velocity was observed to depend on the direction of stirring with respect to the magnetic field which polarizes the atom. The direct comparison with Landau's criterion is not possible since the impurity is not weakly coupled to the BEC, but the anisotropy is clearly there. It is shown to be determined by the magnetic trap, and not the shape of the BEC (right panels in Fig. 1)

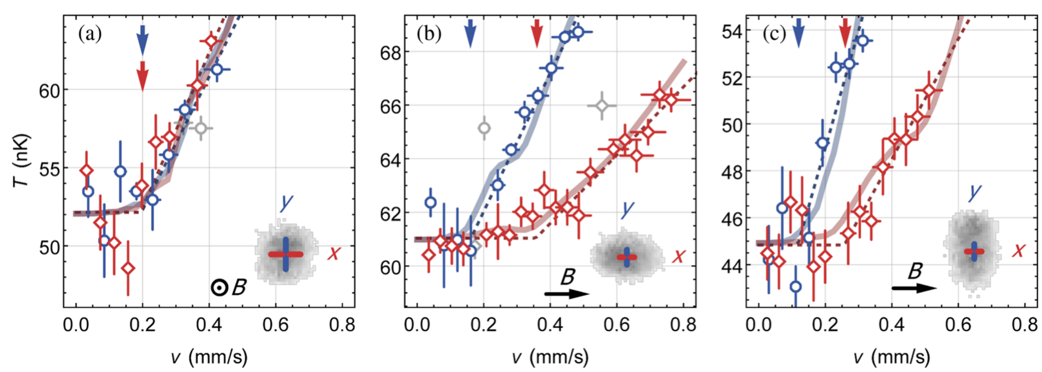

Figure 1: Anisotropy of the critical velocity for superfluidity in a dipolar BEC. When the field is out of the plane, the dissipation does not depend on the motion since dipole-dipole interactions are isotropic in this plane. When the field is along the plane, there is a clear anisotropy. From Wenzel *et al.*, Phys. Rev. Lett. 121, 030401 (2018).