Ultracold atoms and quantum simulators
Problem set 3

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1 Effective range

In this problem we will consider the first correction to the low-energy approximation seen in the lecture: $\lim_{k\to 0} \tan\delta_0 = -ka + ??$

1) We recall the partial wave expansion for the scattering states:

$$\psi_k(r) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^m(\theta,\phi) \frac{u_{k,l,m}(r)}{r}. \tag{1}$$

What is the 1D Schroedinger equation that the u_{kl} satisfy (we consider only $m = 0$ as seen in the lecture). Why is the boundary condition $u_{k,l}(0) = 0$?

2) We consider here low energy scattering by a short-range (range b) potential, scattering is in s-wave only, so we consider only the $l = 0$ term.
Show that for large r ($rb \gg 1$), using the asymptotic form of the partial wave expansion of $\psi_k(r)$, one can write

$$u_{k,0}(r) \propto \sin(kr + \delta_0(k)) \tag{2}$$

3) We rename $u_{k,0}$ as u_k in the following, the free wave function v_k is defined as

$$v_k(r) = \frac{\sin(kr + \delta_0(k))}{\sin(\delta_0(k))}, \tag{3}$$

In the large r limit we have $\lim_{r\gg b} v_k(r) - u_k(r) = 0$.
$v_k(r)$ satisfies the following free Schroedinger equation

$$v_k''(r) + k^2 v_k(r) = 0 \tag{4}$$

From def. (3) the boundary condition for v_k is $v_k(0) = 1$.
What is $\lim_{k\to 0} \frac{d}{dr} v_k(r)$? Conclude that $v_0(r) = 1 - \frac{r}{a}$.

4) Summaize the four Schroedinger equations and the four boundary conditions for $r = 0$ for $u_k(r)$, $v_k(r)$, $u_0(r)$, and $v_0(r)$.

5) Show that

$$\frac{d}{dr}\left[u_k u_0' - u_0 u_k' - v_k v_0' + v_0 v_k'\right] = k^2(u_k u_0 - v_k v_0). \tag{5}$$

6) Show that

$$k\cot(\delta_0) = -\frac{1}{a} + k^2 \int_0^\infty (v_k(r)v_0(r) - u_k(r)u_0(r))dr \tag{6}$$

7) For small enough k we can assume that $\frac{\hbar^2 k^2}{2\mu} \ll V(r)$ when $r < b$. Show finally that the s-wave scattering amplitude takes the form

$$f_0(k) \underset{k\to 0}{\simeq} \frac{1}{-\frac{1}{a} + \frac{1}{2}r_e k^2 - ik}, \tag{7}$$

where $r_e = 2\int_0^\infty (v_0^2(r) - u_0^2(r))dr$.
r_e is called the effective range, it gives the first correction to the scattering when departing from the zero-energy limit.

2 Scattering length for a square well

We consider here a simplified interaction potential: an attractive square well of size b and depth $-\frac{\hbar^2 k_0^2}{2\mu}$:

$$V(r) = \begin{cases} -\frac{\hbar^2 k_0^2}{2\mu} & r \leqslant b \\ 0 & r > b \end{cases} \tag{8}$$

We consider once again only the s-wave scattering part to deduce the s-wave scattering length a. We use the expansion (1) and write once again $u_k = u_{k,0}$.

1) What is the Schroedinger equation verified by u_k for $r \leqslant b$? And for $r > b$?

2) Show that the solutions are :

$$r > b: \quad u_k(r) = A\,\sin(kr + \delta_0) \tag{9}$$
$$r \leqslant b: \quad u_k(r) = A'\,\sin(\kappa r), \tag{10}$$

with $\kappa^2 = k_0^2 + k^2$.

3) Show that

$$k\cot(kb + \delta_0) = \kappa\cot(\kappa b) \tag{11}$$

4) Show then that

$$\tan\delta_0 = \frac{k\tan(\kappa b) - \kappa\tan(kb)}{\kappa + k\tan(\kappa b)\tan(kb)} \tag{12}$$

6) Finally conclude that the s-wave scattering length reads:

$$a = r_0\left(1 - \frac{\tan(k_0 b)}{k_0 b}\right) \tag{13}$$

We thus find that the scattering length undergoes a series of divergences, when $k_0 b = \frac{\pi}{2}(2n+1)$, so the scattering cross-section varies very strongly even though the scattering potential varies smoothly.